

# Asymptotic predictions using short-time data in oscillating billiards

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Particle motion in a smoothly oscillating non-integrable billiard is known to result in unbounded energy growth. Though the asymptotic energy growth rate of an ensemble of particles in an oscillating chaotic billiard is known to be quadratic, there are no estimates available for smoothly oscillating pseudo-integrable billiards. The energy growth rate in such systems is so slow that it is very hard to predict the asymptotic rates from finite time computations. In this paper, a method is proposed to estimate the asymptotic energy growth rate in a system by using short-time data. The idea is applied to the case of an oscillating pseudo-integrable system, and it is shown that the asymptotic energy growth rate in such systems could be exponential.

## I. INTRODUCTION

For particle motion governed by an autonomous Hamiltonian, the total energy remains constant. If we now add a small time-dependent smooth perturbation, the total energy is no longer independent of time. For such systems, the energy can remain bounded or grow unboundedly with time. For 1d autonomous Hamiltonians, one can show using the KAM theorem [1, 2] that the energy will remain bounded under the influence of a small time-dependent perturbation. Though KAM theorem is applicable to 2d and higher dimensional integrable systems too, the energy growth in such systems can still be unbounded due to a phenomenon known as Arnold diffusion [3]. However, this process is exponentially slow and hard to observe in practical situations. For the case of chaotic Hamiltonians, it was recently shown that the energy can grow unboundedly, at a much faster rate compared to Arnold diffusion, if a small time-dependent perturbation is added to the system [4].

One could carry out a similar analysis for an important class of non-smooth Hamiltonians, namely, the dynamical billiard. A billiard is a dynamical system in which a particle moves in straight lines within a region bounded by rigid boundaries, and undergoes specular reflections on collision with the boundary [5–7]. Particle motion in oscillating billiards has been studied in the context of Fermi acceleration [8–10]. The concept of Fermi acceleration has found immense applications in areas like collisional heating in plasma RF sheaths [11] and models of nuclear fission [12]. There also have been several studies of the Fermi accelerator from a quantum mechanical viewpoint [13, 14]. For the 1d Fermi-Ulam model, it has been proved, using the KAM theorem, that the particle energy remains bounded [15]. For the 2d case, it has been numerically shown that energy can grow unboundedly in a smoothly oscillating billiard if the frozen system is chaotic [16–20]. A theory has also been proposed to explain this observation [21].

In between completely integrable and completely chaotic systems, there is an interesting class of pseudo-integrable systems. Such systems are non-integrable but are not chaotic (all Lyapunov exponents are zero). Liouville-Arnold theorem [22] states that if a Hamiltonian system with  $n$  degrees of freedom has  $n$  independent integrals in involution, then it can be integrated by quadratures. However, this notion of integrability is only local. There are systems which have the required number of independent integrals in involution, but do not have global action-angle variables [23, 24]. And this is because the manifold corresponding to the particle motion in these systems is not diffeomorphic to a sphere or a torus, but to a surface of genus greater than one. Rational polygons belong to this interesting class of pseudo-integrable systems [25–27]. A natural question arises regarding the energy growth of a particle if a small time-dependent smooth perturbation is added to such systems. Recently, it has been numerically shown that a particle experiences unbounded energy growth in smoothly oscillating versions of pseudo-integrable billiards [28].

For smoothly oscillating chaotic billiards, it has been numerically observed that the rate of energy growth is quadratic in time [17]. And this observation is also supported by theoretical arguments [21]. However, for the case of simply connected pseudo-integrable systems, there are no known estimates for the rate of energy growth. This paper is the first attempt to give some numerical estimate for the asymptotic energy growth rate in such systems.

Carrying out numerical computations for energy growth in oscillating billiards is computationally intensive. And the lack of accuracy after a certain number of collisions puts a limit on the total time for which the simulation can be carried out. This can have important implications for estimating the asymptotic energy growth rate of particles in a given system. Basically, if the growth rate up to a certain time has been observed to be quadratic, there is no

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guarantee that a higher growth rate will not be observed if the computations could be carried out for a longer time. Thus, it is very important to have some indirect way of predicting the asymptotic growth rate based on short-time data. This paper aims at proposing one such method.

## II. OSCILLATING BILLIARDS

Consider a 2d billiard with an oscillating boundary. The phase space variables are  $\{x_n, y_n; u_n, v_n; t_n\}$  where  $n$  denotes the number of collisions with the oscillating boundary. The particle energy is  $E_n = u_n^2 + v_n^2$ . In general,  $u_n$  and  $v_n$  are not constant. For the sake of simplicity, let us consider a billiard whose oscillating boundary is just a straight line and moves in a direction normal to itself. Let the wall velocity at the instant of the  $n$ th collision be  $w_n$  and always along the  $y$ -axis. The change of velocity of the particle in the  $n$ th collision with the oscillating boundary is

$$\Delta v_n = 2w_n \quad (1)$$

The time interval between two collisions with the oscillating boundary is

$$\Delta t_n = \frac{l_n}{\sqrt{E_n}} \quad (2)$$

where  $l_n$  is the distance travelled by the particle in between the  $n$ th and  $(n-1)$ th collision. As particle energy grows unboundedly ( $E_n \rightarrow \infty$ ), the time-interval between collisions decays to zero ( $\Delta t_n \rightarrow 0$ ), and in this limit, we have

$$\frac{dv}{dt} = \sqrt{E} \frac{2w}{l} \quad (3)$$

Substituting  $v = \sqrt{E} \cos \phi$ , we get

$$\frac{dE}{dt} = E \left[ 2 \tan \phi \frac{d\phi}{dt} + \frac{4w}{l \cos \phi} \right] \quad (4)$$

Now, what we are interested in is the growth rate averaged over the time period of oscillations of the base. Time averaging Eq. (4), we get

$$\frac{d\bar{E}}{dt} = \bar{E} \left\langle 2 \tan \phi \frac{d\phi}{dt} + \frac{4w}{l \cos \phi} \right\rangle_T \quad (5)$$

where  $T = 2\pi/\omega$  is the time period of oscillations of the billiard. For the 1d Fermi-Ulam model,  $\cos \phi = 1$ , and  $l$  is a known periodic function if  $E$  is high enough. Thus, in this case, the average of the right hand side of Eq. (5) is zero and there is no net acceleration of the particle. However, for a general 2d billiard,  $l$  and  $\cos \phi$  change randomly, and if the average,  $\left\langle 2 \tan \phi \frac{d\phi}{dt} + \frac{4w}{l \cos \phi} \right\rangle_T > 0$ , then this leads to an unbounded growth of energy. If the long time average,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left[ 2 \tan \phi \frac{d\phi}{dt} + \frac{4w}{l \cos \phi} \right] dt = R > 0 \quad (6)$$

then the particle also experiences exponentially fast growth of energy,  $\bar{E} = E_0 e^{Rt}$ . However, for a given system, if  $R$  is very low, then it is hard to see the asymptotic exponential growth numerically. This is mainly because numerically accurate computations can be done only for a finite time, which may be too short to observe the exponential energy growth in some systems. In this paper, a way of estimating asymptotic growth rates using short-time data is proposed.

## III. MAKING ASYMPTOTIC PREDICTIONS

The most general equation for evolution of energy of an ensemble of particles in any system is

$$\frac{dE}{dt} = f(E, t) \quad (7)$$

where  $f$  is an arbitrary function of  $E, t$ . The method proposed in this paper is applicable to a sub-class of Eq. (7), namely, those equations where  $f(E, t)$  is variable separable,

$$\frac{dE}{dt} = f(E, t) = g(E) h(t) \quad (8)$$

where  $g(\cdot)$  and  $h(\cdot)$  are two arbitrary functions of their arguments. As can be seen from Eq. (5), the oscillating billiard certainly belongs to this class of systems. If both the functions  $g, h$  are known analytically, then one can surely go ahead and do rigorous analysis. However, in certain situations like particle motion in oscillating billiards, the function,  $h(t)$ , has terms that change in a way that is random for all practical purposes. In such situations, one cannot do any rigorous analysis for predicting the system behavior. Also, as mentioned earlier, one also cannot do computations for very long times due to limits on the accuracy.

The basic idea proposed in this paper is that measuring the dependence of short-time growth rates on the initial energy of the system can give us some indication of asymptotic behavior. For various asymptotic energy growth rates, it can be shown, using Eq. (8), that the dependence of the linear part of the growth rate on initial energy,  $E_0$ , must be

$$\begin{aligned} \text{Linear growth rate} &: E = E_0 + \alpha t \\ \text{Quadratic growth rate} &: E = E_0 + \beta E_0^{0.5} t + \dots \\ \text{Exponential growth rate} &: E = E_0 + \gamma E_0 t + \dots \end{aligned} \quad (9)$$

where  $\alpha, \beta, \gamma$  are arbitrary constants. Thus, the idea is to carry out the simulations till a time for which the growth rate is predominantly linear and then, using Eq. (9), we can say that

1. If a plot of  $(E - E_0)$  is independent of  $E_0$ , the asymptotic growth rate is linear
2. If a plot of  $(E - E_0) / E_0^{0.5}$  is independent of  $E_0$ , the asymptotic growth rate is quadratic
3. If a plot of  $(E - E_0) / E_0$  is independent of  $E_0$ , the asymptotic growth rate is exponential

Now, we use this idea to estimate the asymptotic energy growth of an ensemble of particles in an oscillating trapezium (see Fig. 1), which is a pseudo-integrable system. For the case of a trapezium with an oscillating base, Fig. 2 shows that  $(E - E_0) / E_0$  is independent of  $E_0$  (approximately), hence indicating that the asymptotic energy growth rate in this billiard could be exponential. In Fig. 3, a plot of  $(E - E_0) / E_0^{0.5}$  can be clearly seen to depend on  $E_0$ . Though the plots in Figs. 2 and 3 are not conclusive, they do provide a reasonable indication of the asymptotic energy growth rate being exponential.

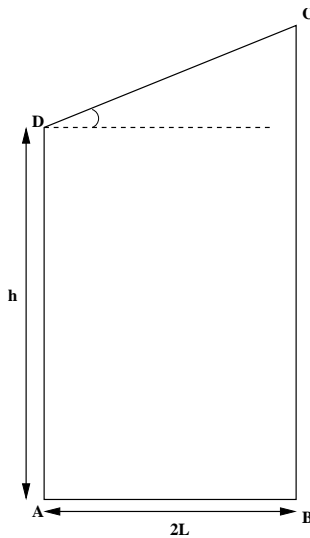


Figure 1: Geometry of the trapezium.

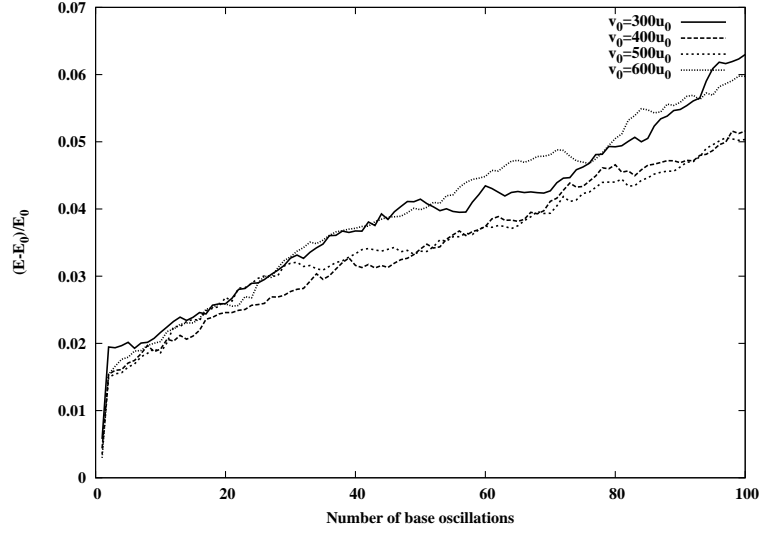


Figure 2: Plot of  $(E - E_0)/E_0$  vs. time for an ensemble of 1000 particles in an oscillating trapezium with parameters:  $L = 0.5$ ,  $h = 4.0$  and  $\angle ADC = 0.5\pi + \pi/18$ . Amplitude of oscillations of the base (AB) was 0.1 with a frequency of  $\omega = 0.1$ , and with an initial horizontal velocity,  $u_0 = 4\omega/\pi$ .

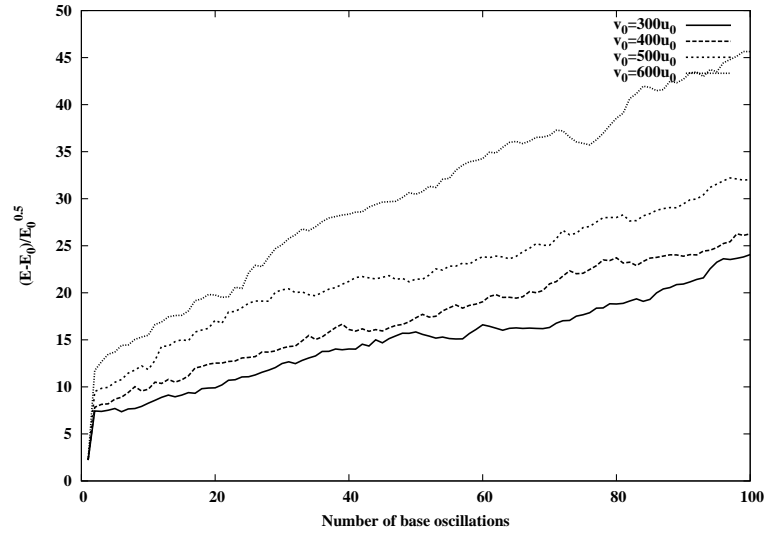


Figure 3: Plot of  $(E - E_0)/E_0^{0.5}$  vs. time for an ensemble of 1000 particles in an oscillating trapezium with the same parameters as in Fig. 2.

## IV. CONCLUSIONS

Though the idea presented in this paper has been applied to the specific case of an oscillating billiard, it is very general in nature. There are many problems in engineering and applied sciences where prediction of asymptotic behavior is of crucial importance. One limitation of the method proposed in this paper is that, using this method, it is hard to differentiate between two growth rates that are not well separated, eg. exponential versus a polynomial of high order. However, I hope that the idea proposed in this paper will provide a starting point in developing more elaborate tools to handle this problem.

## Acknowledgments

I acknowledge support of the Israel Science Foundation and the Minerva Foundation. I would also like to thank Vered Rom-Kedar and Dmitry Turaev for interesting discussions.

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